ISRAEL JOURNAL OF MATHEMATICS 96 (1996), 267–279

THE IRREDUCIBLE CHARACTERS OF THE LIE SUPERALGEBRAS OF TYPE $A(n,m)$ AND FILTRATIONS OF THEIR KAC MODULES

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ABSTRACT

An explicit description of the contra variant induced filtration of the Kac modules of weights which are sl(n) sl(m)) trivial leads to a recursion formula for the irreducible characters of the Lie superalgebras of type sl(n,m).

Introduction

In [5] Kac invented the superversion of the Verma module, which is called now the Kac module. In this paper we study the case $g = \mathrm{sl}(n,m)$, $g = g_0 + g_1$; $g_0 \cong \mathrm{sl}(n)\oplus\mathrm{sl}(m)\oplus c(\tau)$, τ is an odd simple coroot. The odd part g_1 equals $g_1^+ \oplus g_1^-$ (subset of upper and lower triangular matrices in an appropriate representation). Let H be the Cartan subalgebra of g.

Let $\varphi \in H^*$ be a dominant weight; the Kac module K_φ is defined as $K_\varphi = \cup(g) \otimes_{\cup(g_0+g_{1+})} V\varphi$, when $V\varphi$ is the simple $\mathrm{sl}(n)\oplus\mathrm{sl}(m)$ module corresponding to $\varphi\,|_{\mathrm{sl}(n)\oplus\mathrm{sl}(m)}$ on which g_1^+ acts trivially and τ acts on the highest vector of V_φ by the scalar $\varphi(\tau)$. $\cup(g)$ denotes the enveloping algebra of (g).

As a cyclic module, $K\varphi$ possesses a unique maximal submodule. Kac proved that any finite-dimensional simple module of g is obtained as a quotient of $K\varphi$

Received May 14, 1995 and in revised form January 31, 1996

(for an appropriate φ), with its maximal submodule. Although quite a lot of work has been done to obtain the construction of the simple quotient above and of the Kac module itself, the complete picture is not yet clear.

As in other situations of indecomposable objects in representation theory, contra variant forms are quite an effective tool in studying Kac modules of $\mathrm{sl}(n,m)$. (con-var forms were used at first in [8].) The first result of this paper is a complete expression of the con-var form on each $K\varphi$ for which φ is $\mathrm{sl}(n)$ ($\mathrm{sl}(m)$) trivial. Following this, the construction of the corresponding simple module is given, and the filtration associated to the con-var form is shown to be semisimple.

The second result is a recursion process for obtaining the con-var form, and the characters of each composition factor in the Kac module of arbitrary dominant weights, based on the first result.

Let us explain briefly how this is done.

In [2] the idea of the con-var form on the one-parameter family $K\varphi + x\tau$, for given dominant weight φ, as a set of functions of x, was developed. The main result is a formula for the con-var form on the Kac modules of the trivial $\mathrm{sl}(n) \oplus \mathrm{sl}(m)$ weights (they admit a one-parameter family). In considering the branching rule of [1], to the case of tensorial weights it seems reasonable that any Kac module of $\mathrm{sl}(n,m)$ can be embedded in a g_0 trivial Kac module of $\mathrm{sl}(n+k, m+k')$ for k, k' large enough. Therefore it inherits its con-var form from the last one, whose con-var form was computed in [2]. Unfortunately this embedding is not clear enough, except in the case when φ is trivial with respect to $\mathrm{sl}(n)$ (or $\mathrm{sl}(m)$ as well). In that case $K\varphi$ can be embedded in the trivial Kac module of $\mathrm{sl}(n+k, m)$ for k large enough.

We will prove a very explicit branching rule whose crucial corollary is that the embedding $K_{\varphi+x\tau} \subseteq K'_{x\tau'}$ does not depend on the parameter x. (Here $K_{x\tau'}$ is the $\mathrm{sl}(n+k) \oplus \mathrm{sl}(m)$ trivial one-parameter family of $\mathrm{sl}(n+k, m)$.) After this step is done one can use the fact that the con-var form in the trivial g_0 Kac module depends on the $\mathrm{sl}(m) + c(\tau)$ isotypical components only and the fact that the reduction is made with respect to the other parameter $n+k \to n$, to determine the con-var form of $K_{\varphi+x\tau}$ in each of its $\mathrm{sl}(m)$ isotypical components up to a constant (dependent on x), to be the same as those of $K'_{x\tau}$ on the appropriate $\mathrm{sl}(m)$ isotypical component. The con-var form on $K_{\varphi+x\tau}$ is obtained by normalization of the norm of its highest vector with respect to the con-var form of $K'_{x\tau'}$, which is, in fact, a polynomial in x.

The next section deals with preliminaries. After the preliminaries we begin with the branching rule in the case of $\mathrm{sl}(n) \subset g_0 \subseteq \mathrm{sl}(n,m)$ trivial weights. Theorem 2.1 and its corollaries express how the family, $K_{\varphi+x\tau}$ for φ as above, is embedded in the one-parameter family $\overline{K}_{x\tau'}$ of $\mathrm{sl}(n+k,m)$ for appropriate k in terms of the $\mathrm{sl}(m)$ isotypical components of $K_{\varphi+x\tau}$ and $\overline{K}_{x\tau}$. Later on we give the construction of $K_{\varphi+x\tau}$. Theorem 2.9 gives the construction of the simple quotient of $K_{\varphi+x\tau}$ for each x.

Theorem 2.10 states that any quotient in the filtration is semisimple.

Section 3 deals with a recursion process. Theorem 3.1 deals with tensor products of $\mathrm{sl}(n,m)$ modules. Proposition 3.2 says that the reduction of some tensor products as in 3.1, when a Kac module is involved, doesn't depend on the continuous parameter in some sense.

Afterward in 3.3–3.5 we give the recursion process in order to obtain the con-var form on an arbitrary Kac module and the characters of each quotient in the filtration.

1. Preliminaries

Let g denote the special Lie superalgebra $\mathrm{sl}(n,m)$. Denote by g_0 the even part of g; g_1 denotes the odd part (see [4]).

We are viewing g as a subspace of $M_{n+m}(C)$, of the matrices with zero supertrace. Now g_0 is the space of matrices, supported on the $n \times n$, $m \times m$ principal blocks, and g_1 is the space of matrices supported on the $n \times m$, $m \times n$ nonprincipal blocks. The notation g_0' refers to the Lie algebra $\mathrm{sl}(n) \oplus \mathrm{sl}(m)$ which is included naturally in g.

The notations $g^{\pm}$ refer to upper (lower) triangular matrices in g, $g_i^{\pm} = g_i \cap g^{\pm}$.

H denotes the Cartan subalgebra which comprise the diagonal matrices in g.

A set of positive coroots is chosen to be $\{h^{ij}/i<j\le n; n<i<j\} \cup \{h^{n,n+1}\}$. In the first set h^{ij} means the diagonal matrix with 1 in the ii place, -1 in the jj place and zero elsewhere; $h^{n,n+1}$ is the diagonal matrix with 1 in the n,n place and in the $n+1,n+1$ place.

The dual of $h^{n,n+1}$ with respect to a simple coroot basis of H includes $h^{n,n+1}$ and is denoted by τ and is associated with a continuous parameter in our studies.

An element $\varphi \in H^*$ is called a **weight**, and is a **dominant weight** if $\varphi(h^{ij}) \in N$ for any $i<j\le n$ or $n<i<j$.

For any dominant weight φ let us define V_φ to be the simple module of g_0

corresponding to $\varphi\left|_{g_0'\cap H}\right.$, with the extra requirement that $h^{n,n+1}$ acts on the highest vector of V_φ as the scalar $\varphi(h^{n,n+1})$.

After the action of g_1^+ is defined to be trivial, the Kac module K_φ is defined as $K_\varphi = \mathsf{U}(g)\otimes_{\mathsf{U}(g_0+g_1^+)} V\varphi$. Here U denotes the enveloping algebra. As a g_0' module $K\varphi \cong \bigwedge(g_1^-)\otimes V^\varphi$, where $\bigwedge(g_1^-)$ is the exterior algebra above g_1^-. It is worth noting that as a g_0' module K_φ does not depend on the value of τ. Hence in our studies we divide the set of dominant weights into sets of one-parameter families, each one of which is determined by a dominant g_0' weight. The importance of the Kac modules comes from the next theorem of Kac.

1.1 Each simple g module of finite dimension is a quotient module of some Kac module, see [4].

A dominant weight φ is called **typical** if $K\varphi$ is irreducible.

1.2 By Ehresman's theorem [7], $\bigwedge g_1^- \cong_{g_0} \oplus U^\eta$, where η goes over the set of the Young diagrams in the $n\times m$ rectangle and U^η is the simple g_0 module obtained by matching of the sl(n) simple module corresponding to η and the sl(m) simple one corresponding to η^*. For each η, the highest vector of U^η is given by $\bigwedge_{ij} e^{ij}$ where the double index goes over the black set in Figure 1.

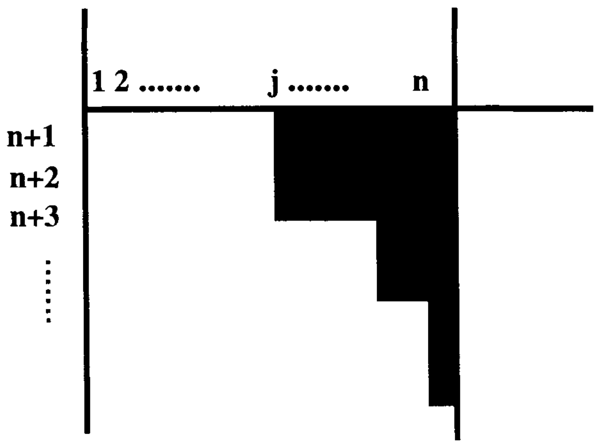

Figure 1.

1.3 As a Corollary of 1.2 we get that knowledge of the g_0' constituents of $K\varphi$ is a matter of reduction of some tensor products of simple g_0' modules.

For any dominant φ the family $K_{\varphi+x\tau}(x\in c)$ possesses a con-var form which depends polynomially on x. It is well known that any g_0' simple module possesses a unique con-var form which is definite, so the dependence of the $\mathrm{sl}(n,m)$ con-var form on x is completely determined by its reduction to the space of g_0' highest vectors in $K_{\varphi+x\tau}$. As $K_{\varphi+x\tau}$ is generated by the highest vector, one can see that $K_{\varphi+x\tau}$ includes a unique maximal submodule which, in fact, coincides with the radical of the con-var form on $K_{\varphi+x\tau}$. See [8] section 4 and [6] lecture 3.

Furthermore, the con-var form on the family $K_{\varphi+x\tau}$ induces filtration on $K_{\varphi+x\tau}$, by taking zeroes of higher orders of the form.

In [2] the con-var form of the family $K_{x\tau}$ (the g_0' trivial family) was computed, as follows.

1.4 By 1.2, $K_{x\tau} \cong_{g_0'} \oplus U^\eta$, the summation running over the Young diagrams η in the rectangle $n \times m$.

On each U^η the con-var form is obtained by the polynomial

$$P_\eta = \prod_{ij}(i + j - 2n - 1 + x).$$

The double index ij runs as in 1.2. For $\eta \neq \eta'$, U^η and $U^{\eta'}$ are orthogonal to each other.

2. The branching rule for $K_{\varphi+x\tau}$ when $\varphi\,|_{\mathrm{sl}(n)}$ is trivial and its explicit construction

First we define precisely the reduction from $\mathrm{sl}(n,m)$ to $\mathrm{sl}(n-1,m)$. Recall that we are thinking of $\mathrm{sl}(n,m)$ as a subspace of $M_{n+m}(\mathbb{C})$ of matrices of zero supertrace. Now, $\mathrm{sl}(n-1,m) \subset \mathrm{sl}(n,m)$ refers to the space spanned by the set

$$\left\{e^{sl}/0 < s, l \leq n + m, s, l \neq n\right\} \cap \mathrm{sl}(n,m)$$

2.1 Theorem (Branching rule): *Let φ be a dominant weight trivial with respect to* $\mathrm{sl}(n)$. *Let Θ be the set of distinct* $\mathrm{sl}(m)$ *highest vectors in the* $\mathrm{sl}(m)$ *module* $\bigwedge(\mathrm{span}\left\{e^{in}/i > n\right\}) \otimes V^\varphi$. *The weights of the elements of Θ can be computed by Young's rule.*

(i) *Each $v \in \Theta$ is a* $\mathrm{sl}((n-1),m)$ *highest vector.*

(ii) $K\varphi + x\tau \cong \bigoplus_{v\in\Theta} \bigwedge(\tilde{g_1^-}) \otimes < v >$, *when* $\tilde{g_1^-} = \mathrm{sl}(n-1,m) \cap g_1^-$ *and* $< v >$ *is the* $\mathrm{sl}(m)$ *simple module generated by v.*

See Figure 2.

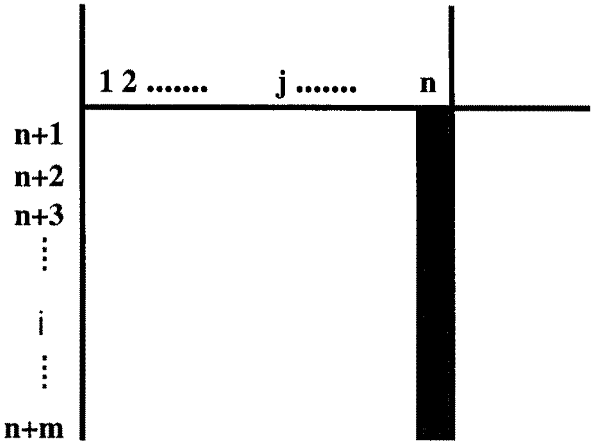

Figure 2.

Proof: (i) One can check that the elements $e^{in}, i > n$ commute with the elements of $\mathrm{sl}(n-1) \subset \mathrm{sl}(n) \subset g_0$. Now since φ is $\mathrm{sl}(n)$ trivial, any $v \in \Theta$ is $\mathrm{sl}(n-1)$ highest vector. Let $i < n, j > n$ and let us compute $e^{ij}v$ for $v \in \Theta$. By definition $v = \sum_s r_s \otimes v_s$, r_s is a monomial in the set $\{e^{in}; i > n\}$ and $v_s \in V_\varphi$. For any $s, e^{ij}(r_s \otimes v_s) = [r_s e^{ij}] \otimes v_s \pm r_s \otimes e^{ij} v_s = [r_s e^{ij}] \otimes v_s$ since e^{ij} acts trivially on V_φ. By definition $[e^{ij} e^{ln}] = \delta_{jl} e^{in}$ for $i < n$. As φ is $\mathrm{sl}(n)$ trivial and $e^{in} \in \mathrm{sl}(n) \subset g_0$ for $i < n, [e^{ij} e^{ln}] \otimes v_s = 0$ for any $l \leq n$. Hence $[e^{ij} r_s] \otimes v_s$ vanishes and v is $\mathrm{sl}(n-1, m)$ highest vector as claimed.

The proof of (ii) comes from the fact that $\bigwedge(g_1^-) \otimes V_\varphi$ is free over $\bigwedge(\tilde{g}_1^-)$ and the use of simple dimensional arguments.

2.2 Remark: The reduction of $K_{\varphi + x\tau}$ under the action of $\mathrm{sl}(n-1, m)$ does not depend on x under the constraint on φ above.

2.3 Remark: The $\mathrm{sl}(n-1, m)$ constituents of $K_{\varphi + x\tau}$ are $\mathrm{sl}(n-1, m)$ Kac modules of highest weights under the same constraint we began with.

2.4 COROLLARY: *Let φ be a $\mathrm{sl}(n, m)$ dominant weight s.t. $\varphi|_{\mathrm{sl}(n)}$ is trivial. There exists k s.t. $K\varphi + x\tau$, $x \in c$ is embedded in the family $\overline{K}_{x\tau}$, $x \in C$, of g_0 trivial Kac modules of $\mathrm{sl}(n + k, m)$. Furthermore, the embedding does not depend on x.*

Proof: By Theorem 2.1 and the Young rule (see [3] p.88, also compare with [1] p.148) one can take k to be as large as the number of columns in a Young diagram corresponding to φ by Schur–Weyl correspondence, for $K_{\varphi+x\tau}$ to appear after k steps of reduction with respect to the first index (n). Using (2.2) and (2.3) successively one can show that the embedding does not depend on x.

We are prepared now to compute the embedding of the family $K_{\varphi+x\tau}$ when φ is sl(n) trivial, within $\overline{K}_{x\tau}$, the g_0' trivial family of sl($n+k, m$).

Since the construction of $\overline{K}_{x\tau}$ is given in terms of Young diagrams, one has to associate a Young diagram to each sl(m) isotypical component of $K_{\varphi+x\tau}$.

There exists a unique Young diagram of k columns (for k big enough) corresponding to φ (as sl(m) weight). We denote it by Y_φ.

For the other sl(m) isotypical components to be treated, assume that $n+k \neq m$ (without loss of generality, because k can be chosen as large as we need). Now let $\gamma \in H$ be an element s.t. $[\gamma l] = (n+k-m)l \neq 0$ for any $l \in g_{\bar{1}}^-$ (here $g = \mathrm{sl}(n+k, m)$). Hence γ acts on $\overline{K}_{x\tau}$ as a grade function. (The grading on $\overline{K}_{x\tau}$ is induced by the grading of $\bigwedge(g_{\bar{1}}^-)$.)

Let Y be a Young diagram in the $n \times m$ rectangle as in 1.2 and let U^Y be its sl(m) simple module. Using γ it is clear that the sl(m)'s components of $U^Y \otimes U^\varphi \subset K_{\varphi+x\tau}$ are placed in the sl(m) components associated with Young diagrams in the $(n+k \times m)$ rectangle of $|Y|+|Y_\varphi|$ cells. (Recall that Y_φ is already chosen.) Here $|\cdot|$ denotes the number of cells.

The next proposition expresses the embedding of the family $K_{\varphi+x\tau}$ in $\overline{K}_{x\tau}$.

2.5 PROPOSITION: *Let φ, Y_φ, k and $\overline{K}_{x\tau}$ be as before. Then:*

(i) *Each Young diagram Y in the $(n+k)\times m$ rectangle is associated with a* sl(m) *isotypical component in $K_{\varphi+x\tau} \subset \overline{K_{x\tau}}$ of multiplicity $\sum_\nu d(\nu) C^\nu_{Y\setminus Y_\varphi}$. The summation runs over the Young diagrams in the $n \times m$ rectangle. $d(\nu)$ is the dimension of the* sl(n) *simple module associated with ν. $C^\nu_{Y\setminus Y_\varphi}$ are the Littlewood–Richardson' symbols, see [3] page 87. All the* sl(m) *isotypical components of $K_{\varphi+x\tau}$ are obtained in this way.*

(ii) *The isotypical component in* (i) *is embedded in the isotypical component of $\overline{K}_{x\tau}$, which we denoted in 1.2 by U^Y.*

(iii) *When x is given, the* sl(n, m) *Kac module above corresponds to the weight $\varphi|_{\mathrm{sl}(m)} + (x-k)\tau$.*

Proof: (i) Follows from Ehresman's theorem (1.2) and the usual treatment with tensor products via Young diagrams.

(ii) Follows from the fact that the reduction is done merely with respect to n, so the sl(m) isotypical components of $K_{\varphi+x\tau}$ should be placed in the same sl(m) isotypical components of $\overline{K}_{x\tau}$. The value of γ determines the grading of the component within $\overline{K}_{x\tau}$. Hence (ii).

(iii) Recall our assumption on Y_φ that it has k columns. Now the gap k, between the value of the continuous parameter on $\overline{K}_{x\tau}$ and on $K_{\varphi+x\tau}$, comes from the fact that the operator $h^{nn+1} \subset$ sl(nm) is transmitted to the operator $h^{n,n+k+1} \in$ sl$(n+k,m)$, together with the fact that $[h^{n,n+k+1}, \prod_{ij\in Y_\varphi} e^{ij}] = -k\prod_{ij\in Y_\varphi} e^{ij}$. $\prod_{ij\in Y_\varphi} e^{ij}$ is the highest vector of the family $K_{\varphi+x\tau}$ within $\overline{K}_{x\tau}$ (see the branching theorem (2.1) and (1.2)).

Now we can give the complete expression for the con-var form of the family $K_{\varphi+x\tau}$ for φ which is sl(n) trivial.

As we mentioned in the preliminaries, it is enough to know the values of the con-var form on the subspaces of g_0's highest vectors in $K_{\varphi+x\tau}$ as a polynomial of x.

From (2.2), the embedding of $K_{\varphi+x\tau}$ does not depend on x, so the g_0' simple modules in $K_{\varphi+x\tau} \subseteq \overline{K}_{x\tau}$ inherit the con-var form of $\overline{K}_{x\tau}$ given in (1.4) directly.

For a Young diagram Y in the $(n+k)\times m$ let U^Y be the associated sl(m) isotypical component of K_φ (on which γ acts as the scalar $|Y|$, and sl(m) acts according to the shape of Y).

2.6 THEOREM: *For given Y_φ, Y and k as before, the con-var form of $K_{\varphi+x\tau}$ reduced to U^Y is determined by the polynomial $\prod_{ij\in Y\setminus Y_\varphi}(i+j-2n-k+x-1)$.*

The range of the multiplication is the black domain in Figure 3.

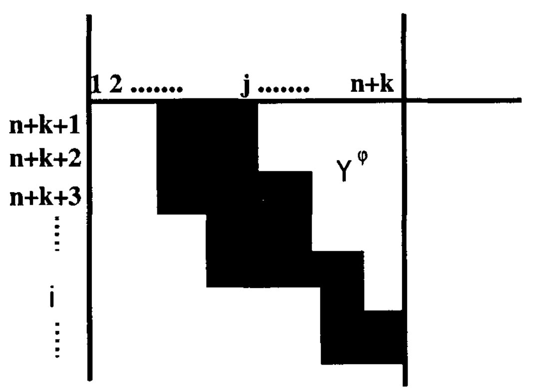

Figure 3.

2.7 Remark: Although k appears in the formula, the polynomial does not depend on the choice of k (which is somehow arbitrary) because k appears as a constant in the definition of j.

Proof: First we mention that by Proposition 2.5 the relevant Y's is our study are those such that $Y \supseteq Y^{\varphi}$, so the claim makes sense. For the formula to be obtained one can use the formula in Theorem 1.4 and the fact that the embedding of $K_{\varphi+x\tau}$ in $\overline{K}_{x\tau}$ does not depend on x. The supplement of k in the formula comes from (iii) of Proposition 2.5. The range on which the indices ij run comes from the normalization of the highest vector's norm of $K_{\varphi+x\tau}$.

2.8 COROLLARY: *The con-var form on $K_{\varphi+x\tau}$ as a function of x is fixed on each isotypical* $\mathrm{sl}(m)$ *component, i.e. the con-var form is definite on each* $\mathrm{sl}(m) \oplus C(\gamma)$ *isotypical component of $K_{\varphi+x\tau}$.*

The next theorem is a direct corollary of Theorem 2.6 and gives the complete construction of the simple $\mathrm{sl}(n,m)$ module of weight $\varphi + x\tau$ as a g_0 module once φ is $\mathrm{sl}(n)$ trivial.

2.9 THEOREM: *Let φ be a dominant* $\mathrm{sl}(m)$ *weight. Let Y_φ be chosen and k the number of columns in Y_φ. Draw Y_φ in the $(n+k, m)$ rectangle. For given x draw the line $x - k + i + j - 2n - 1 = 0$ on the $n + k \times m$ rectangle. Let (i', j') be the lower point of the intersection of the line with Y_φ (Figure 4).*

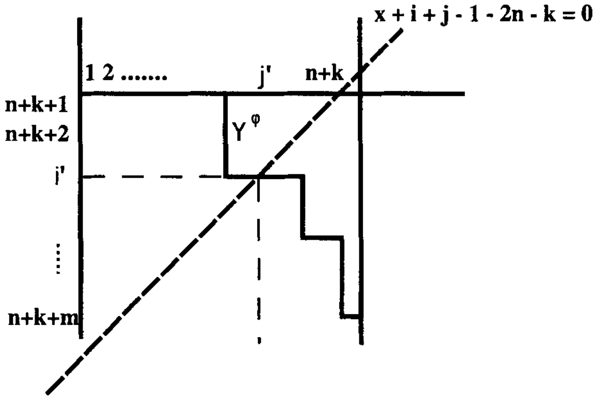

Figure 4.

Then the simple module of the weight $\varphi + x\tau$ as a g_0 module is equal to $\oplus U^Y$ once Y runs over the Young diagrams in the hook which is defined by the point

(i',j') in the $(n+k)\times m$ rectangle, and U^Y is the $\mathrm{sl}(m)+c\gamma$ *isotypical component in K_φ corresponding to Y.*

Proof: As we mentioned in the preliminaries, the maximal submodule of $K_{\varphi+x\tau}$ coincides with the radical of the con-var form of $K_{\varphi+x\tau}$, so by 2.6 it includes, precisely, all the $\mathrm{sl}(m)$ isotypical components corresponding to Young diagrams which include at least one point on the line which is not in Y_φ. The simple quotient includes the other isotypical components in K_φ. This ends the proof.

As we mentioned in the preliminaries, the con-var form on $K_{\varphi+x\tau}$ as polynomials in x induces a filtration on $K_{\varphi+x\tau}$ as follows: to any u,v in K_φ the value of the con-var form in $K_{\varphi+x\tau}$ is a polynomial in t. Denote it by $(vu)_t$. For given x, define for any natural i:

$$V_i^x = \{v/(v,u)_t \text{ is divided by } (t-x)^i \text{ for any } u \text{ in } K_\varphi\}.$$

Clearly $K_{\varphi+x\tau} = V_0^x \supset V_1^x \supset V_2^x \cdots$ is a sequence of $\mathrm{sl}(n,m)$ submodules.

2.10 THEOREM: *Let φ be a dominant* $\mathrm{sl}(n)$ *trivial weight. Then for any given x and i, V_i^x/V_{i+1}^x is semisimple.*

Proof: It is clear that V_i^x/V_{i+1}^x inherits a nondegenerate con-var form from $K_{\varphi+x\tau}$.

Assume $U \subseteq V_i^x/V_{i+1}^x$ is a submodule. We claim that $U^\perp \subseteq V_i^x/V_{i+1}^x$ is a linear complement of U. $U^\perp$ has the right dimension, because the form on V_i^x/V_{i+1}^x is nondegenerate. So assume that $U\cap U^\perp \neq 0$. As $U\cap U^\perp$ is a $\mathrm{sl}(n,m)$ submodule it contains an isotypical $\mathrm{sl}(m)$ component, which is, by our assumption, self-orthogonal. However, the con-var form is definite on each $\mathrm{sl}(m)$ isotypical component in V_i^x (see 2.8), hence the contradiction, because distinct isotypical components are orthogonal to each other.

3. The construction of $K\varphi + x\tau$ for arbitrary φ. A recursive approach

We need some information on tensor products. Let Y be a Young diagram in the n,m hook which includes the $n\times m$ rectangle. Let K_Y be the Kac module corresponding to Y and V_r be the simple module corresponding to one row of length r.

3.1 THEOREM: *$K_Y \otimes V_r = \oplus K_{\bar{Y}}$, summed over all $K_{\bar{Y}}$ s.t. $Y_i \leq \bar{Y}_i \leq Y_{i+1}$, and $\sum_i \bar{Y}_i - Y_i = r$.*

Proof: This is an immediate Corollary of the Littlewood–Richardson rule (the same as in the classical case).

Since our goal is to get the con-var form on $K_{\varphi+x\tau}$ in terms of polynomials in x, the x's dependence on the reduction in 3.1 is acute. Since for a large enough integer x, $K_{\varphi+x\tau}$ is tensorial and typical, we can assume without loss of generality that φ is tensorial and typical and denote the corresponding Young diagram by Y_φ.

3.2 PROPOSITION: *The reduction of $K_{\varphi+x\tau} \otimes V_r$ doesn't depend on x.*

Proof: At first we deal with $K_{\varphi+x\tau}\otimes V_1$. As $\mathrm{sl}(n)\oplus\mathrm{sl}(m)$ module V_1 is isomorphic to $U \oplus T$, when $U(T)$ is the natural representation of $\mathrm{sl}(n)(\mathrm{sl}(m))$. Let V_φ be the $\mathrm{sl}(n) \oplus \mathrm{sl}(m)$ simple module corresponding to $\varphi|_{\mathrm{sl}(n)\oplus\mathrm{sl}(m)}$. Clearly $V_\varphi \cong V_{\varphi|_{\mathrm{sl}(n)}} \otimes V_{\varphi|\,\mathrm{sl}(m)} \equiv V_n \otimes V_m$.

Let v be a $\mathrm{sl}(n)$ highest vector in $V_n \otimes U$, and t be the highest vector in T; then $v \otimes t$ is $\mathrm{sl}(n,m)$ highest vector. Hence we have a direct summand of $K_{\varphi+x\tau} \otimes V_1$ as: $\oplus \wedge (g_1^-)\otimes < v \otimes t >$ summed over all v as above. Clearly this direct summand of $K_{\varphi+x\tau} \otimes V_1$ does not depend on x. To complete the reduction of $K_{\varphi+x\tau} \otimes V_1$, recall the element $\gamma = \bigwedge_{i>n,j<n} e^{ij}$ (see [5]) which commutes with $\mathrm{sl}(n)\oplus\mathrm{sl}(m)$. Now let v be a $\mathrm{sl}(m)$ lowest vector in $V_m \otimes T$ and u the $\mathrm{sl}(m)$ lowest vector in V_n. It is clear that $\gamma \otimes v \otimes u$ is $\mathrm{sl}(n,m)$ lowest vector. Hence $\oplus \bigwedge(g_1^+) \otimes \gamma\otimes < v \otimes u >$, summed over all v as above, is a direct summand of $K_{\varphi+x\tau} \otimes V_1$, which does not depend on x. By Theorem 3.1 one can see that: $K_{\varphi+x\tau} \otimes V_1 = (\oplus \bigwedge(g_{1+}) \otimes \gamma\otimes < v \otimes u >) \oplus (\bigwedge(g_{1-})\otimes < v \otimes u >)$. In the first summand v runs over all the $\mathrm{sl}(m)$ lowest vectors in $V_n \otimes U$. In the second v runs over the $\mathrm{sl}(n)$ highest vectors in $V_m \otimes T$. This is a reduction which doesn't depend on x.

Let r be arbitrary. V_r is a direct summand of $V_1^{\otimes r}$. Hence $K_{\varphi+x\tau} \otimes V_r$ is a direct summand of $K_{\varphi+x\tau} \otimes V_1^{\otimes r}$. Clearly this reduction doesn't depend on x. The proposition is proved now by using the particular case, $K_{\varphi+x\tau} \otimes V_1$ successively r times.

3.3 With 3.1 and 3.2 one can set a recursion formula for the characters of the factors of the filtration of $K_{\varphi+x\tau}$ for each φ and x. One can take any order on the set of Young diagrams such that Y is the minimal element of the set of Young diagrams which have appeared in 3.1; for instance, the lexicographic order on the length of rows up from the bottom.

Let Y be a Young diagram in the n, m hook including the $n \times m$ rectangle and let φ_Y be its weight. Let $\chi^i_Y(x)$ be the character of V^x_i (see 2.10). Now if Y is included in the n strip one can use the explicit formula we developed in Section 2. Otherwise, let Y' be the Young diagram which is accepted from Y by dropping the last row of Y. By our assumption on Y, Y' includes the $n \times m$ rectangle.

3.4 THEOREM:

(i) $\chi^i_Y(x) = \chi^{i-IY}_{Y'}(x)\chi_r - \sum_{\bar{Y}>Y} \chi^{i-I\bar{Y}}_{\bar{Y}}(x)$ *summed over* $\bar{Y}$ *as in 3.1.*

(ii) $I\bar{Y} = \min\{i|(\chi^i_{Y'}(x)\chi_r)_{\varphi_{\bar{Y}}+x\tau} \neq \sum_{\hat{Y}>\bar{Y}}(\chi^{i-I\hat{Y}}_{\hat{Y}}(x))_{\varphi_{\bar{Y}+x\tau}}\}$.

Proof: On $K_{\varphi+x\tau} \otimes V_r$ one can define a con-var form by the rule $(h \otimes v \cdot h' \otimes v') = (hh')(v \cdot v')$ to $h, h' \in K_{\varphi+x\tau}$, $v, v' \in V_r$. The forms on the right-hand side are the con-var forms on $K_{\varphi+x\tau}$ and V_r. Now use Theorem 3.1 in splitting $\chi^i_Y(x)\chi_r$ between the Kac modules that appeared in $K_{\varphi_{Y'}} \otimes V_r$ as constituents.

3.5 Remark: The boundary points of the recursion process we described are Young diagrams of the type we deal with in Section 2 (see Figure 5).

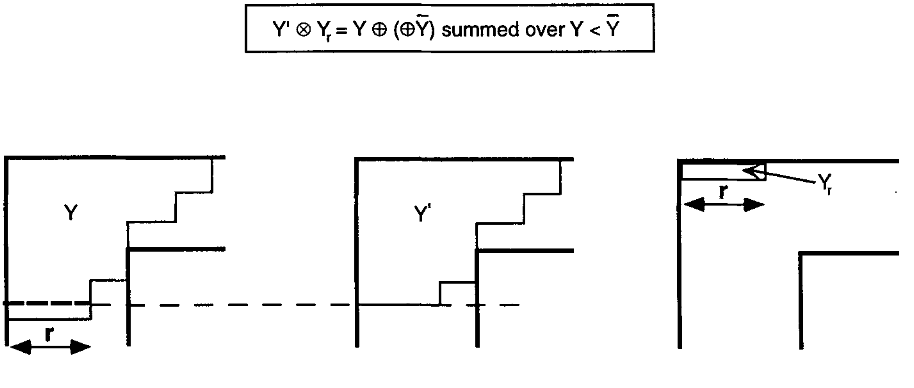

Figure 5.

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